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Bi-Complementarity and Duality: A Framework in Nonlinear Equilibria with Applications to the Contact Problem of Elastoplastic Beam Theory*

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Nonlinear complementarity problems and variational inequalities in nonlinear equilibrium problems are studied within a unified framework. Based on the generalized Rockafellar–Tonti diagram, a bi-complementarity problem with both internal and external nonlinear complementarity conditions is proposed. A general duality theory in variational inequality is established and the Mosco dual variational inequality has been generalized to the nonsmooth systems. In order to study the frictional contact problem of beam theory, a two-dimensional elastoplastic beam model is proposed. The external complementarity condition provides the free boundary of contact region, while the internal complementarity condition gives the interface of the elastic–plastic regions. Our results shown that in nonsmooth equilibrium problems, the dual approaches are much easier than the primal problems. © 1998 Academic Press

Key Words: complementarity problems; duality; variational inequality; nonsmooth analysis; contact problem; beam theory.

1. INTRODUCTION

Complementarity theory has become a rich source of inspiration in both mathematical and engineering sciences. Since the nonlinear complementarity problem (NCP) was introduced in Cottle's [4] Ph.D. dissertation, many different kinds of complementarity problems have been proposed, and the theory has been extended and generalized in various directions to

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study a wide class of problems arising in optimization and control, mechanics, operation research, fluid dynamics, economics, and transportation equilibrium. Several monographs have documented the historical details about the origin of these problems and their evolution (cf., e.g., [5, 22, 28]). The comprehensive, up-to-date treatment of the NCP in finite dimensions and the extensive documentation of applications in engineering and equilibrium modeling were given recently in survey articles [9, 33].

It is known that the complementarity problems are special cases of the variational inequality [9]. Variational methods and direct approaches for unilateral variational problems and free boundary value problems have been studied extensively during the last 30 years. Here we only mention the well-known books by Duvaut and Lions [7], Kinderlehrer and Stampacchia [25], Glowinski, Lions, and Tremolieres [19], Panagiotopoulos [31, 32], Rodrigues [36], Kikuchi and Oden [24], Hlavacek *et al*. [21], and Friedman [10]. From the point of view of optimization, the direct methods for solving primal variational inequality problems provide only upper bound approaches of the problems. Meanwhile, the solutions of dual variational inequalities will give lower bound approaches of the problems. For nonlinear, nonsmooth systems, dual approaches have more favorable properties. Recent research results shown that some fully nonlinear equilibrium problems in finite deformation theory can be solved by using a nonlinear dual transformation method (see [15]). The dual variational inequality was first studied by Mosco [29] in 1972. Since then many papers have been published in this field (see, for example, [1, 20, 26, 27, 34]). Recently, the duality in geometrically nonlinear (i.e., weakly nonlinear) variational inequality was studied for von Karman plates [39] and for nonlinear elastic beam theory [13, 14]. Actually, by introducing a so-called superpotential, the variational inequalities in most conservative systems are equivalent to certain nonsmooth variational problems (cf., e.g., [31, 32]). The fields of nonsmooth optimization and of inequality mechanics have experienced significant development in recent decades. Complicated phenomena in engineering, mechanics, economics, and many other fields require the consideration of nondifferentiability for their accurate modeling. A novel approach to this important area was presented recently by Dem'yanov *et al*. [6].

The purpose of the present paper is to establish a unified framework and duality theory for nonlinear, nonsmooth complementarity problems. Based on the generalized Rockaffelar–Tonti diagram, a nonlinear bi-complementarity problem with both internal and external complementarity conditions is proposed, and a classification for various complementarity conditions is given in Section 2. In Section 3, a duality theory is established for variational inequality problems. The Mosco dual variational inequality is generalized to the nonsmooth systems with internal complementarity condition.

The contact problems in elasticity have been studied extensively from both theoretical and numerical aspects (cf., e.g., [24, 32]). It is known that Timoshenko beam theory is an extension of classical beam theory and allows for the effect of transverse shear deformation by relaxing the normality assumption. But in this theory, the shear deformation does not vary in the lateral beam direction, i.e., the plane section remains plane (but not necessarily normal to the longitudinal axis) after deformation. This beam model is not well adapted for studying frictional contact problems. Recently, an extended Timoshenko beam model has been established to take account of shear variation in the lateral direction [16]. In the present paper, this extended beam model is generalized to elastoplastic beam theory with different displacement variables. A second order partial differential inequality system is established in Section 4. The external complementarity condition defines the free boundary of the contact region, while the internal complementarity condition gives rise to the interface between the elastic and plastic domains. Our results shown that in nonsmooth mechanics, dual problems are much easier than primal problems.

2. CLASSIFICATION OF COMPLEMENTARITY PROBLEMS

Let $(\mathscr{V}, \mathscr{V}^*)$ and $(\mathscr{E}, \mathscr{E}^*)$ be two pairs of real vector spaces, finite or infinite dimensional, in duality with respect to certain bilinear functions $(*, *)$ and $\langle *, * \rangle$, respectively. In mathematical physics, if we call $\mathscr{V}$ the "configuration" space, its dual space $\mathscr{V}^*$ should be the "source" space. $\mathscr{E}$ and its dual space $\mathscr{E}^*$ are called intermediate variable spaces (see [30, 35, 37, 38]). Let the geometrical operator Λ be a continuous linear transformation from $\mathscr{V}$ to $\mathscr{E}$. Its adjoint $\Lambda^* : \mathscr{E}^* \to \mathscr{V}^*$ is defined by

$$\langle \Lambda u, e^* \rangle = (u, \Lambda^* e^*) \qquad \forall u \in \mathscr{V}, e^* \in \mathscr{E}^*. \tag{1}$$

According to Rockafellar [35] and Tonti [38], three equations can be defined in mathematical physics:

$$\begin{aligned} &\textit{Geometrical Equation}: && e = \Lambda u, \\ &\textit{Constitutive Equation}: && e^* = C(e), \\ &\textit{Equilibrium Equation}: && u^* = \Lambda^* e^*, \end{aligned} \tag{2}$$

where the constitutive mapping $C : \mathscr{E} \to \mathscr{E}^*$ could be a nonsmooth, monotone operator. The problem is called *physically nonlinear* (*or strongly* *nonlinear*) if the constitutive operator C is nonlinear. In terms of u only, this physical nonlinear equilibrium problem can be written in the fundamental form

$$\Lambda^* C(\Lambda u) = u^*. \tag{3}$$

If C is a linear symmetric operator, this fundamental form can be written as

$$Au = \Lambda^* C \Lambda u = u^*,$$

where $A := \Lambda^* C \Lambda$ is a symmetric operator. In the celebrated textbook by Strang [37], this nice symmetrical structure can be seen from continuous theories to discrete systems. The problem is called *geometrically nonlinear* (*or weakly nonlinear*) if the geometrical operator Λ is nonlinear. In this case, the geometrical equation and constitutive equation are the same as in (2). But the equilibrium equation should be

$$u^* = \Lambda_t^*(u) e^*, \tag{4}$$

where $\Lambda_t^*: \mathscr{E}^* \to \mathscr{U}^*$ is the adjoint operator of Λ_t, the Gâteaux derivative of $e = \Lambda u$ (see [17]). The duality theory for geometrically nonlinear variational problems as well as applications in finite deformation mechanics have been discussed in a series of papers (see [16–18]).

In complementarity problems, two kinds of general complementarity conditions can be classified on the paired spaces $(\mathscr{U}, \mathscr{U}^*)$ and $(\mathscr{E}, \mathscr{E}^*)$, respectively:

(i) External complementarity condition,

$$\begin{gathered} B(u) \geq 0, \qquad B^*(u^*) \leq 0, \qquad (B(u), B^*(u^*)) = 0 \\ \forall (u, u^*) \in \mathscr{U} \times \mathscr{U}^*, \end{gathered} \tag{5}$$

where $B: \mathscr{U} \to \mathscr{U}$ and $B^*: \mathscr{U}^* \to \mathscr{U}^*$ could be linear or nonlinear operators. In the present paper, we consider only the affine operators

$$B(u) = u(x) - \psi(x), \qquad B^*(u^*) = -u^*(x) + \bar{u}^*(x). \tag{6}$$

$\psi \in \mathscr{U}$ and $\bar{u}^* \in \mathscr{U}^*$ are given functions. In physics and engineering applications, condition (5) usually controls the boundary complementarity condition. For example, in contact problems, where ψ is the obstacle function, $\bar{u}^*$ is the given external load. This complementarity condition gives the free boundary of the contact region (see Section 4).

(ii) Internal complementarity condition,

$$g(e) \geq 0, \qquad g^*(e^*) \leq 0, \qquad g(e)^T g^*(e^*) = 0 \qquad \forall (e, e^*) \in \mathscr{E} \times \mathscr{E}^*, \tag{7}$$

where $g\colon \mathscr{E} \to \mathbf{R}^n$ and $g^*\colon \mathscr{E}^* \to \mathbf{R}^n$ are vector-valued functions. In elastoplasticity, this complementarity condition models the interface of elastic–plastic regions. Figure 1 shows the inner relations of the framework of the bi-complementarity problems.

Let $\mathscr{V}_a$ be a feasible (or kinetically admissible configuration) space, in which the essential boundary conditions are prescribed. Then the *bi-*

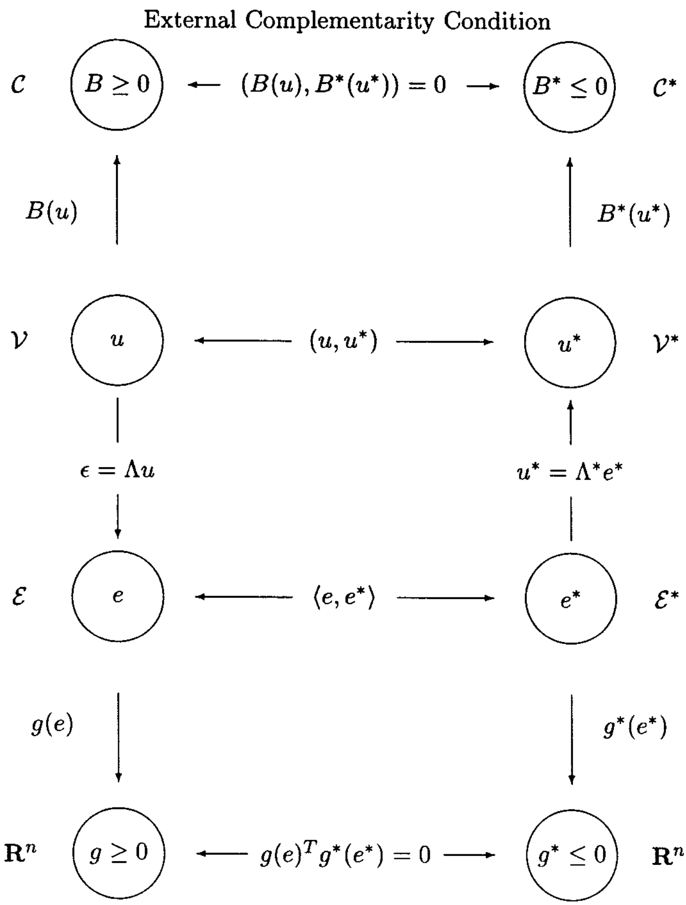

FIG. 1. Framework for bi-complementarity problems in geometrically linear systems.

complementarity problem ((BCP) for short) for geometrical linear systems can be proposed as follows:

Problem 1 (BCP). For the given functions $\psi(x)$ and $\bar{u}^*(x)$, find $u \in \mathscr{U}_a$ such that

(1) $\Lambda^* C(\Lambda u) = u^*$,

(2) $B(u) \geq 0$, $B^*(u^*) \leq 0$, $(B(u), B^*(u^*)) = 0$,

(3) $g(\Lambda u) \geq 0$, $g^*(C(\Lambda u)) \leq 0$, $g(\Lambda u)^T g^*(C(\Lambda u)) = 0$.

In continuum mechanics, the configuration variable u is a displacement vector; $e = \Lambda u$ is a strain tensor. Its dual variable $e^* = \sigma$ should be a stress tensor. For elastoplastic material, the vector-valued function $g(e)$ is a so-called plastic flow factor; its dual function $g^*(\sigma)$ is the plastic yield function, which is a convex function. In the elastic region Ω_{e}, we have $g(e) \leq 0$, $g^*(\sigma) < 0$, while in the plastic region Ω_{p}, $g(e) > 0$, $g^*(\sigma) = 0$. The internal complementarity condition (7) gives the interface $\Gamma_{\mathrm{ep}} = \Omega_{\mathrm{e}} \cap \Omega_{\mathrm{p}}$ of the elastic−plastic regions. For the so-called *nonassociated plastic materials*, such as soil, rocks, etc., the plastic flow factor and the yield function are independent on the constitutive equation. However, for those so-called *associated plastic media*, like engineering metals and composite materials, the complementarity conditions depend on the constitutive equation. In this case, we can introduce a plastic superpotential $W: \mathscr{E} \to \mathbf{R}^{\oplus} := \mathbf{R} \cup \{+\infty\}$, such that the constitutive equation and the internal complementarity conditions are equivalent to the subdifferential form

$$e^* \in \partial W(e), \tag{8}$$

where $\partial W: \mathscr{E} \to \mathscr{E}^*$ is a set-valued mapping:

$$\partial W(e) := \{e^* \in \mathscr{E}^* \mid \langle e^*, \epsilon - e \rangle \leq W(\epsilon) - W(e) \ \forall \epsilon \in \mathscr{E}\}.$$

The variable $e^* \in \partial W(e)$ is called the subgradient (see [8]). If $W: \mathscr{E} \to \mathbf{R}$ is Gâteaux differentiable, then $\partial W(e) = \{DW(e)\}$. Here DW denotes the Gâteaux derivative of W with respect to e.

The conjugate function of W can be given by the Legendre−Fenchel transformation:

$$W^*(e^*) = \sup_{e \in \mathscr{E}} \{\langle e^*, e \rangle - W(e)\}. \tag{9}$$

Obviously, $W^*: \mathscr{E}^* \to \mathbf{R}^{\oplus}$ is convex, l.s.c. If $W: \mathscr{E} \to \mathbf{R}^{\oplus}$ is convex, then

$$e^* \in \partial W(e) \quad \Leftrightarrow \quad e \in \partial W^*(e^*) \quad \Leftrightarrow \quad W(e) + W^*(e^*) = \langle e^*, e \rangle.$$

For example, in the extension of an elastoplastic bar, the internal complementarity condition and the constitutive equation between the strain $e = \Lambda u = du/dx$ $(x \in [0, L])$ and the stress $\sigma = e^*$ can be written as

$$\sigma = Ce \qquad \text{if } g(e) = e - e_b < 0,$$

$$g(e) \geq 0, \qquad g^*(\sigma) \leq 0, \qquad g(e)g^*(\sigma) = 0,$$

where e_b is a positive constant; the real-valued function $g^*(\sigma) = \sigma - \sigma_b$ is the plastic yield function, $\sigma_b = Ce_b$. By introducing the step function δ,

$$\delta(g) = \begin{cases} 1, & \text{if } g \geq 0, \\ 0, & \text{if } g < 0, \end{cases}$$

the superpotential energy function can be defined as

$$W(e) = \int_0^L \left[\frac{1}{2} Ce^2 \delta^c(g) + \frac{1}{2} Ce_b^2 \delta(g) + \sigma_b g(e) \delta(g) \right] dx,$$

where $\delta^c(g) := 1 - \delta(g)$. The conjugate superpotential energy is

$$\begin{aligned} W^*(\sigma) &= \sup_e \left\{ \int_0^L \sigma e \, dx - W(e) \right\} \\ &= \int_0^L \left[\frac{1}{2} C^{-1} \sigma^2 \delta^c(g^*) + \frac{1}{2} C^{-1} \sigma_b^2 \delta(g^*) \right] dx \\ &\quad + \begin{cases} 0, & \text{if } g^*(\sigma) \leq 0, \\ +\infty, & \text{if } g^*(\sigma) > 0. \end{cases} \end{aligned}$$

So the constitutive equation and the internal complementarity condition for this associated elastoplastic bar can be simply written in the equivalent subdifferential inclusions,

$$\sigma \in \partial W(e) \quad \Leftrightarrow \quad e \in \partial W^*(\sigma).$$

The idea of associated plasticity can be generalized to complementarity problems. In conservative systems, if we can define a superpotential W such that its subdifferential is equivalent to the constitutive equation and internal complementarity conditions, then the so-called *associated bi-complementarity problem* (ACP) can be proposed as follows:

Problem 2 (ACP). For the given ψ and $\bar{u}^*$, find $u \in \mathscr{V}_a$ such that

$$u^* \in \Lambda^* \partial W(\Lambda u), \tag{10}$$

$$B(u) \geq 0, \qquad B^*(u^*) \leq 0, \qquad \langle B(u), B^*(u^*) \rangle = 0. \tag{11}$$

If the system has the internal complementarity condition only, then Problem 2 (ACP) is reduced to the *associated internal complementarity problem* (AICP);

Problem 3 (AICP). For a given source $\bar{u}^* \in \mathscr{V}^*$, find $u \in \mathscr{V}_a$ such that

$$\bar{u}^* \in \Lambda^* \partial W(\Lambda u). \tag{12}$$

In convex analysis, this nonlinear equilibrium problem can be considered as a Euler–Lagrange equation of a primal variational problem (see [8]).

3. VARIATIONAL INEQUALITY AND DUALITY THEORY

Let $\mathscr{C}$ be a closed convex subset of $\mathscr{V}$:

$$\mathscr{C} := \{ v \in \mathscr{V} \mid B(v) \geq 0 \}. \tag{13}$$

If $B(v) = v$, $\mathscr{C}$ is a convex cone with vertex at the origin. The dual space $\mathscr{C}^*$ can be defined as

$$\mathscr{C}^* := \{ v^* \in \mathscr{V}^* \mid (v^*, B(v)) \leq 0 \ \forall v \in \mathscr{C} \}. \tag{14}$$

The indicator function $\Psi_{\mathscr{C}}$ of the convex set $\mathscr{C}$ is a convex, l.s.c. function from $\mathscr{V}$ to $\mathbf{R}^{\oplus}$:

$$\Psi_{\mathscr{C}}(v) = \begin{cases} 0, & \text{if } v \in \mathscr{C}, \\ +\infty, & \text{otherwise}. \end{cases} \tag{15}$$

It is easy to prove the equivalent relations

$$u^* \in \partial \Psi_{\mathscr{C}}(u) \quad \Leftrightarrow \quad u^* \leq 0, \qquad B(u) \geq 0, \qquad (u^*, B(u)) = 0. \tag{16}$$

In many boundary value problems with given external source $\bar{u}^* \in \mathscr{V}^*$, we can define a concave, u.s.c. function $F\colon \mathscr{V} \to \mathbf{R}^{\ominus} := \mathbf{R} \cup \{-\infty\}$ such that

$$\bar{u}^* \in \bar{\partial} F(u), \tag{17}$$

where $\bar{\partial} F\colon \mathscr{V} \to \mathscr{V}^*$ is the so-called *superdifferential* (see [2]), defined as $\bar{\partial} F = -\partial(-F)$. For example, if $\mathscr{V}_a$ is a feasible subspace of $\mathscr{V}$, in which the essential boundary conditions are prescribed, then F can be defined as

$$F(u) = (\bar{u}^*, u) - \Psi_{\mathscr{V}_a} = \begin{cases} (\bar{u}^*, u), & \text{if } u \in \mathscr{V}_a, \\ -\infty, & \text{otherwise}. \end{cases} \tag{18}$$

So the total potential energy $P\colon \mathscr{C} \to \mathbf{R}^{\oplus}$ in nonlinear equilibrium problems can be defined as

$$P(u) = W(\Lambda u) - F(u). \tag{19}$$

The *primal variational problem* (PVP) can be proposed as follows: Find $u \in \mathscr{C}$ such that

$$(\mathrm{PVP}) \qquad P(u) = \inf P(v) \qquad \forall v \in \mathscr{C}. \tag{20}$$

We can prove that this problem is equivalent to the following *primal variational inequality problem* (PVI):

$$(\mathrm{PVI}) \qquad (\Lambda^* \partial W(\Lambda u), v - u) \geq F(v) - F(u) \qquad \forall v \in \mathscr{C}. \tag{21}$$

If P is Gâteaux differentiable on $\mathscr{V}$, the directional derivative of P at u in the direction $v \in \mathscr{V}$ is

$$\delta P(u; v) = \lim_{t \to 0^+} \frac{P(u + tv) - P(u)}{t} = (DP(u), v),$$

where $DP(u)\colon \mathscr{V} \to \mathscr{V}^*$ is the Gâteaux derivative of P at u defined by

$$DP(u) = \Lambda^* DW(\Lambda u) - DF(u).$$

Then the PVI can be written as

$$(\mathrm{PVI}') \qquad (DP(u), v - u) \geq 0 \qquad \forall v \in \mathscr{C}. \tag{22}$$

Since $\mathscr{C}$ is a convex cone, $\mathscr{C}^*$ is its polar cone; it is obvious that the associated bicomplementarity problem (ACP) can be written as the *primal complementarity problem* (PCP)

$$(\mathrm{PCP}) \qquad u \in \mathscr{C}, \qquad \partial P(u) \in \mathscr{C}^*, \qquad (\partial P(u), B(u)) = 0. \tag{23}$$

LEMMA 1. *If* $\Lambda\colon \mathscr{V} \to \mathscr{E}$ *is a linear, continuous operator,* $P\colon \mathscr{C} \to \mathbf{R}^{\oplus}$ *is convex, l.s.c., then*

$$(\mathrm{PVP}) \Leftrightarrow (\mathrm{PVI}) \Leftrightarrow (\mathrm{PCP}). \tag{24}$$

The proof of this lemma is similar to the proof of Theorem 1 given below.

Direct methods for solving nonsmooth variational problems with bicomplementarity conditions are very difficult. In this section, we are interested in establishing the dual problems. To find the conjugate function of $P\colon \mathscr{C} \to \mathbf{R}^{\oplus}$, we should extend its domain $\mathscr{C}$ to $\mathscr{V}$ by defining

$$J(u, e) = W(e) - F(u) + \Psi_{\mathscr{C}}(u) = W(e) - F_c(u),$$

where $F_c(u) = F(u) - \Psi_{\mathscr{C}}(u)$. It is obvious that on $\mathscr{C}$, $P(u) = J(u, \Lambda u)$. For geometrical linear operator Λ, the conjugate function $J^*: \mathscr{V}^* \times \mathscr{E}^* \to \mathbf{R}^{\ominus}$ of J can be easily given as (cf., e.g., [8])

$$\begin{aligned} J^*(u^*, e^*) &= \inf_{u \in \mathscr{V}} \inf_{e \in \mathscr{E}} \{ (u^*, u) + \langle -e^*, e \rangle + J(u, e) \} \\ &= F_c^*(\Lambda^* e^*) - W^*(e^*), \end{aligned}$$

where $W^*: \mathscr{E}^* \to \mathbf{R}^{\oplus}$ is given by (9). $F_c^*: \mathscr{V}^* \to \mathbf{R}^{\ominus}$ is given by

$$F_c^*(u^*) = \inf_{u \in \mathscr{V}} \{ (u^*, u) - F(u) + \Psi_{\mathscr{C}}(u) \}.$$

Let $u = \phi + \psi$. Then we have

$$\begin{aligned} F_c^*(u^*) &= \inf_{u \in \mathscr{C} \cap \mathscr{V}_a} \{ (u^*, u) - (\bar{u}^*, u) \} \\ &= \inf_{\phi \geq 0} \{ (u^* - \bar{u}^*, \psi) + (u^* - \bar{u}^*, \phi) \} \\ &= \begin{cases} F^*(u^*), & \text{if } -u^* + \bar{u}^* \leq 0, \\ -\infty, & \text{if } -u^* + \bar{u}^* > 0, \end{cases} \end{aligned} \tag{25}$$

where $F^*(u^*) = (u^* - \bar{u}^*, \psi)$. Let $\mathscr{E}_a^* \subset \mathscr{E}^*$ be a *dual feasible space*, in which the natural boundary conditions are prescribed, and $\mathscr{D}^*$ is a convex subset of $\mathscr{E}_a^*$:

$$\mathscr{D}^* := \{ e^* \in \mathscr{E}_a^* \mid B^*(\Lambda^* e^*) \leq 0 \}.$$

On $\mathscr{D}^*$, the conjugate function of P is then obtained as

$$P^*(e^*) = J^*(\Lambda^* e^*, e^*) = F^*(\Lambda^* e^*) - W^*(e^*). \tag{26}$$

From the theory of convex analysis, $P^*: \mathscr{D}^* \to \mathbf{R}^{\ominus}$ is always concave, u.s.c. So the *dual variational problem* (DVP) can be proposed as: To find $e^* \in \mathscr{D}^*$ such that

$$(\mathrm{DVP}) \qquad P^*(e^*) = \sup P(\epsilon^*) \qquad \forall \epsilon^* \in \mathscr{D}^*. \tag{27}$$

The associated *dual variational inequality problem* is to find $e^* \in \mathscr{D}^*$ such that

$$\langle \partial W^*(e^*), \epsilon^* - e^* \rangle \geq F^*(\Lambda^* \epsilon^*) - F^*(\Lambda^* e^*) \qquad \forall \epsilon \in \mathscr{D}^*. \tag{28}$$

If P^* is Gâteaux differentiable on $\mathscr{D}^*$, then

$$\delta P^*(e^*; \epsilon^*) = \lim_{t \to 0^+} \frac{P^*(e^* + t\epsilon^*) - P^*(e^*)}{t} = \langle DP^*(e^*), \epsilon^* \rangle,$$

where the Gâteaux derivative DP^* is given as

$$DP^*(e^*) = \Lambda DF^*(\Lambda^* e^*) - DW^*(e^*).$$

Then the (DVI) can be written as

$$(\mathrm{DVI}') \qquad \langle DP^*(e^*), \epsilon^* - e^* \rangle \leq 0 \qquad \forall \epsilon^* \in \mathscr{D}^*. \tag{29}$$

If the inverse operator Λ^{-1} of Λ exists, we can let

$$\mathscr{D} := \left\{ e \in \mathscr{E} \mid \left(\Lambda^{-1} e, B^*(\Lambda^* e^*) \right) \geq 0 \ \forall e^* \in \mathscr{D}^* \right\}. \tag{30}$$

The *dual complementarity problem* (DCP) then can be formulated as

(DCP)

$$e^* \in \mathscr{D}^*, \qquad \bar{\partial} P^*(e^*) \in \mathscr{D}, \qquad \left(\Lambda^{-1} \bar{\partial} P^*(e^*), B^*(\Lambda^* e^*) \right) = 0. \tag{31}$$

Furthermore, if the problem has only the external complementarity condition, W: $\mathscr{E} \to \mathbf{R}$ is convex, differentiable such that $e = \partial W^*(e^*) = C^{-1} e^*$, then the dual variational inequality (DVI′) can be reduced into the form

$$(A' u^*, v^* - u^*) \geq F^*(v^*) - F^*(u^*) \qquad \forall v^* \in \Lambda^* \circ \mathscr{D}^*. \tag{32}$$

This is the well-known Mosco dual variational inequality [29], where

$$A' = -A^{-1} - : v^* \to -A^{-1}(-v^*)$$

is a formal inverse of $A = \Lambda^* C \Lambda$. Unfortunately, in infinite-dimensional problems, where Λ is a gradient-like operator, to find the inverse $A^{-1} = \Lambda^{-1} C^{-1} \Lambda^{*-1}$ is almost impossible. This is the reason why the applications of Mosco's dual variational inequality are limited. But in dual variational inequalities (DVI) and (DVI′), instead of v^*, we use $\epsilon^* \in \mathscr{E}^*$ as the variational argument, so these dual problems are very easy to formulate.

If the system has the associated internal complementarity condition only, the dual complementarity problem can be simply given as

$$(\mathrm{DCP}') \qquad e^* \in \mathscr{E}_a^*, \qquad 0 \in \bar{\partial} P^*(e^*). \tag{33}$$

In many engineering problems, this dual problem is much easier than the primal problem.

THEOREM 1. *If P: $\mathscr{C} \to \mathbf{R}^{\oplus}$ is convex and Gâteaux differentiable, then the primal problems* (PVP), (PVI), (PCP) *and the dual problems* (DVP), (DVI), (DVI′) *are equivalent in the sense that they have same solutions set and*

$$\inf_{u \in \mathscr{C}} P(u) = \sup_{e^* \in \mathscr{D}^*} P^*(e^*). \tag{34}$$

Proof. Let us prove (DVI) $\Leftrightarrow$ (DVP) first. Since $W^*: \mathscr{E}^* \to \mathbf{R}^{\oplus}$ is a convex, l.s.c., one has

$$W^*(\epsilon^*) - W^*(e^*) \geq \langle \partial W^*(e^*), \epsilon^* - e^* \rangle \qquad \forall \epsilon^*, e^* \in \mathscr{E}^*.$$

If $e^* \in \mathscr{D}^*$ is a solution of (DVI), then

$$P^*(e^*) - P^*(\epsilon^*) \geq \langle \partial W^*(e^*), \epsilon^* - e^* \rangle - F^*(\Lambda^*\epsilon^*) + F^*(\Lambda^* e^*) \geq 0$$
$$\forall \epsilon^* \in \mathscr{D}^*.$$

This shows that (DVI) $\Rightarrow$ (DVP). Since $\mathscr{D}^*$ is convex, for any given $\theta \in [0,1]$, one has

$$e^* \in \mathscr{D}^*, \qquad \epsilon^* \in \mathscr{D}^* \quad \Rightarrow \quad \theta e^* + (1-\theta)\epsilon^* \in \mathscr{D}^*$$

From the DVP, we should have

$$P^*(e^*) \geq P^*(\theta\epsilon^* + (1-\theta)e^*) = P^*(e^* + \theta(\epsilon^* - e^*)) \qquad \forall \theta \geq 0,$$

i.e.,

$$\frac{1}{\theta}\left[P^*(e^* + \theta(\epsilon^* - e^*)) - P^*(e^*)\right] \leq 0 \qquad \forall \theta \geq 0.$$

Taking $\theta \to 0^+$, we obtain

$$0 \geq \lim_{\theta \to 0^+} \frac{1}{\theta}\left[P^*(e^* + \theta(\epsilon^* - e^*)) - P^*(e^*)\right] = \langle DP^*(e^*), \epsilon^* - e^* \rangle.$$

This shows that (DVP) $\Rightarrow$ (DVI$'$). Since $F^*: \mathscr{D}^* \to \mathbf{R}^{\ominus}$ is concave, u.s.c., and Gâteaux differentiable, one has

$$\begin{aligned} F^*(\Lambda^*\epsilon^*) - F^*(\Lambda^* e^*) &\leq \left(\bar{\partial} F^*(\Lambda^* e^*), \Lambda^*\epsilon^* - \Lambda^* e\right) \\ &= \langle \Lambda\bar{\partial} F^*(\Lambda^* e^*), \epsilon^* - e^* \rangle \qquad \forall \epsilon^* \in \mathscr{D}^*. \end{aligned}$$

So it is easy to find that (DVI$'$) $\Leftrightarrow$ (DVI). Actually, if $P^*: \mathscr{D}^* \to \mathbf{R}^{\ominus}$ is strictly concave and $\mathscr{D}^*$ is a closed subset of a reflexive Banach space, then each dual problem has a unique solution.

Now we are going to prove that the dual solution of (DVP) solves (PCP). Introducing the Lagrange multiplier $u \in \mathscr{V}_a$ to relax the constraint in (DVP), the Lagrangian $L(u, e^*)$: $\mathscr{V}_a \times \mathscr{E}_a^* \to \mathbf{R}^{\ominus}$ associated with DVP can be given by

$$L(u, e^*) = P^*(e^*) - (B(u), B^*(\Lambda^* e^*)). \tag{35}$$

So the DVP is equivalent to the saddle-point problem

$$\inf_{v \in \mathscr{V}_a} \sup_{\epsilon^* \in \mathscr{E}_a^*} L(v, \epsilon^*). \tag{36}$$

The Kuhn–Tucker optimality conditions for this saddle-point problem are

$$0 \in \bar{\partial}_{e^*} L(u, e^*), \qquad B(u) \geq 0, \qquad B^*(\Lambda^* e^*) \leq 0, \\ (B(u), B^*(\Lambda^* e^*)) = 0. \tag{37}$$

Since $W\colon \mathscr{E} \to \mathbf{R}^{\oplus}$ is convex, l.s.c., the partial superdifferential inclusion $0 \in \bar{\partial}_{e^*} L$ gives the subdifferential constitutive relations

$$\Lambda u \in \partial W^*(e^*) \quad \Leftrightarrow \quad e^* \in \partial W(\Lambda u).$$

This shows that the solution of the saddle-point problem (36) solves the complementarity problem (PCP).

For any given $v \in \mathscr{C}$ and $\epsilon^* \in \mathscr{D}^*$, using the Legendre transformation $F^*(\Lambda^* \epsilon^*) = (\Lambda^* \epsilon^*, v) - F(v)$ to replace F^* in P^*, the Lagrangian $L(v, \epsilon^*)$ associated with the dual variational problem (DVP) can be written as

$$L(v, \epsilon^*) = \langle \Lambda v, \epsilon^* \rangle - W^*(\epsilon^*) - F(v). \tag{38}$$

Since $W(e)$ is convex, $W^{**} = W$, it is easy to find that $\sup_{\epsilon^*} L(\epsilon^*, v) = P(v)$. On the other hand, if $\epsilon^* \in \mathscr{D}^*$, $\inf_v L(\epsilon^*, v) = P^*(\epsilon^*)$. Since $L\colon \mathscr{C} \times \mathscr{D}^* \to \mathbf{R}$ is a saddle-point functional, we have

$$\inf_v P(v) = \inf_v \sup_{\epsilon^*} L(v, \epsilon^*) = \sup_{\epsilon^*} \inf_v L(v, \epsilon^*) = \sup_{\epsilon^*} P^*(\epsilon^*) \\ \forall (v, \epsilon^*) \in \mathscr{C} \times \mathscr{D}^*.$$

So Eq. (34) is proved. ∎

Let us now demonstrate how the above scheme fits in with the finite-dimensional nonlinear programming and NCP. Let $\mathscr{V} = \mathscr{V}^* = \mathbf{R}^n$, $\mathscr{E} = \mathscr{E}^* = \mathbf{R}^m$, with the standard coordinatewise partial ordering. Consider the general global optimization problem:

$$\text{(P)} \qquad \min f(u) \text{ s.t. } u \in \mathscr{C}, \qquad g(\Lambda u) \geq 0,$$

where $\mathscr{C} \subset \mathbf{R}^n$ is a nonempty convex cone, $f\colon \mathscr{C} \to \mathbf{R}$ is l.s.c., $\Lambda \in \mathbf{R}^{m \times n}$ is a matrix, and $g\colon \mathscr{E} \to \mathbf{R}^m$ is componentwise l.s.c. on $\mathscr{E}$. To reformulate this nonlinear constrained optimization problem in the model form (PVP), i.e., $P(u) = W(\Lambda u) - F(u)$, we need only set

$$F(u) = -f(x) - \Psi_{\mathscr{C}}(u),$$

$$W(e) = \begin{cases} 0 & \text{if } g(e) \geq 0, \\ +\infty & \text{otherwise}. \end{cases}$$

Then the finite-dimensional optimization problem (P) can be written as a unconstrained nonsmooth optimization problem:

$$P(u) = W(\Lambda u) - F(u) \to \min \qquad \forall u \in \mathbf{R}^n.$$

Consider the linear programming case, where

$$\mathscr{C} = \{u \in \mathbf{R}^n \mid u \geq 0\},$$

$$f(u) = (u, \bar{u}^*), \qquad g(\Lambda u) = \Lambda u - b.$$

For fixed $\bar{u}^* \in \mathbf{R}^n$ and $b \in \mathbf{R}^m$, (P) is a linear programming

$$(\mathrm{PVP_{lin}}) \qquad \min(u, \bar{u}^*) \text{ s.t. } u \geq 0, \qquad g(\Lambda u) = \Lambda u - b \geq 0.$$

The conjugate functions in this elementary case may be calculated at once from (9) and (25) as

$$W^*(e^*) = \begin{cases} \langle b, e^* \rangle, & \text{if } e^* \leq 0, \\ +\infty & \text{otherwise}, \end{cases}$$

$$F^*(u^*) = \begin{cases} 0 & \text{if } u^* + \bar{u}^* \geq 0, \\ -\infty & \text{otherwise}. \end{cases}$$

So the dual problem is

$(\mathrm{DVP_{lin}})$

$$\max\langle -b, \epsilon^* \rangle \text{ s.t. } \epsilon^* \leq 0, \qquad g^*(\epsilon^*) = -\Lambda^*\epsilon^* - \bar{u}^* \leq 0.$$

The Lagrangian (38) in this case is

$$L(v, \epsilon^*) = (v, \bar{u}^*) + \langle \Lambda v - b, \epsilon^* \rangle \qquad \forall v \geq 0,\ \epsilon^* \leq 0,$$

i.e., the Lagrangian multiplier of the primal problem should be the solution of the dual variational problem. The Kuhn−Tucker conditions for this problem are

$$u \geq 0, \qquad \Lambda^* e^* + \bar{u}^* \geq 0, \qquad (u, \Lambda^* e^* + \bar{u}^*) = 0,$$

$$e^* \leq 0, \qquad \Lambda u - b \geq 0, \qquad \langle \Lambda u - b, e^* \rangle = 0.$$

So this is a bi-complementarity problem.

4. CONTACT PROBLEM OF ELASTOPLASTIC BEAM

As a typical example, we will use the extended elastic−perfectly-plastic beam theory developed recently [13, 14] to illustrate the primal and dual bi-complementarity problems. Let us consider a elastic−perfectly-plastic beam possibly in contact with a rigid obstacle, which is described by a strictly concave function $\psi(x)$ (see Fig. 2).

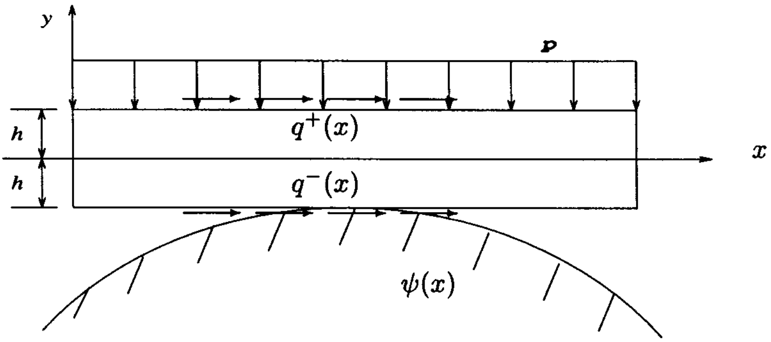

FIG. 2. Frictional contact problem of elastoplastic beam.

Suppose that the beam in the x-y plane is a rectangle $\Omega = \{(x, y) \in \mathbf{R}^2 \mid 0 \leq x \leq L, -h \leq y \leq h\}$. The beam is subjected to a given distributed load $\bar{\mathbf{p}} = (\bar{q}^+(x), \bar{p}(x))^T$ on the top surface $y = h$ (see Fig. 2). On the bottom surface, the beam is subjected to a frictional shear force $\bar{q}^-(x)$, which is unknown until the problem is solved. Displacement of material point (x, y) in the beam is described by the vector

$$\mathbf{u}(x, y) = \begin{pmatrix} u(x, y) \\ w(x) \end{pmatrix}, \qquad (x, y) \in \Omega.$$

The first component describes the horizontal displacement of the material point, while the second describes the displacement of the middle axis, which coincides with the x axis in the equilibrium state.

The general strain vector is given by the geometrical equation

$$\mathbf{e} = \begin{pmatrix} \epsilon \\ \gamma \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial w}{\partial x} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial}{\partial x} & 0 \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} u(x, y) \\ w(x) \end{pmatrix} = \Lambda \mathbf{u}. \quad (39)$$

In this problem, the geometrical operator is a linear differential operator:

$$\Lambda := \begin{pmatrix} \dfrac{\partial}{\partial x} & 0 \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{pmatrix}. \quad (40)$$

Let $\mathscr{V}$ be the general displacement space,

$$\mathscr{V} := \left\{ \mathbf{u} = \begin{pmatrix} u(x,y) \\ w(x) \end{pmatrix} \mid u(x,y) \in \mathscr{H}^1(\Omega), w(x) \in \mathscr{H}^1[0,L] \right\},$$

where $\mathscr{H}^1 = \mathscr{W}^{2,1}$ is the standard Sobolev space. The source space $\mathscr{V}^* = \mathscr{F}$ is a *general force space*:

$$\mathscr{F} := \left\{ \mathbf{p} = \begin{pmatrix} q(x) \\ p(x) \end{pmatrix} \mid q(x), p(x) \in \mathscr{L}^2[0,L] \right\}.$$

The bilinear form $(*,*)$: $\mathscr{F} \times \mathscr{U} \to \mathbf{R}$ can be written as

$$(\mathbf{p}, \mathbf{u}) = \int_0^L q^+(x) u(x,h)\, dx + \int_0^L q^-(x) u(x,-h)\, dx$$

$$+ \int_0^L p(x) w(x)\, dx.$$

The *general strain space* $\mathscr{E}$ and its dual space $\mathscr{E}^* = \mathscr{S}$, the *general stress space*, in this problem are

$$\mathscr{E} := \left\{ \mathbf{e} = \begin{pmatrix} \epsilon(x,y) \\ \gamma(x,y) \end{pmatrix} \middle| \epsilon(x,y), \gamma(x,y) \in \mathscr{L}^2(\Omega) \right\},$$

$$\mathscr{S} := \left\{ \mathbf{s} = \begin{pmatrix} \sigma(x,y) \\ \tau(x,y) \end{pmatrix} \middle| \sigma(x,y), \tau(x,y) \in \mathscr{L}^2(\Omega) \right\}.$$

The bilinear form $\langle\ ,\ \rangle$: $\mathscr{S} \times \mathscr{E} \to \mathbf{R}$, i.e.,

$$\langle \mathbf{s}, \mathbf{e} \rangle = \langle \sigma, \epsilon \rangle + \langle \tau, \gamma \rangle = \int_\Omega (\sigma\epsilon + \tau\gamma)\, d\Omega,$$

encompasses the duality relationship between $\mathscr{S}$ and $\mathscr{E}$.

Using the Gauss–Green theorem, for any given $\mathbf{u} = (u, w)^T$ such that $\mathbf{e} = \Lambda\mathbf{u}$, the duality relation (1) in this problem is

$$\langle \mathbf{s}, \Lambda\mathbf{u} \rangle = (\Lambda^*\mathbf{s}, \mathbf{u}), \tag{41}$$

where Λ^*: $\mathscr{S} \to \mathscr{F}$ is the operator adjoint to Λ relative to $\langle *,* \rangle$, $(*,*)$ defined by

$$(\Lambda^*\mathbf{s}, \mathbf{u}) = -\int_\Omega \frac{\partial\tau}{\partial x} w\, d\Omega - \int_\Omega \left(\frac{\partial\sigma}{\partial x} + \frac{\partial\tau}{\partial y} \right) u\, d\Omega$$

$$+ \int_{-h}^h \sigma n_x u\, dy \mid_{x=0,L} + \int_{-h}^h \tau n_x w\, dy \mid_{x=0,L}$$

$$+ \int_0^L \tau n_y u\, dx \mid_{y=\pm h}.$$

Thus Λ^* involves the "interior" operator in the domain Ω defined by

$$\Lambda^*_{\text{int}} = \int_{-h}^{h} \begin{pmatrix} -\dfrac{\partial}{\partial x} & -\dfrac{\partial}{\partial y} \\ 0 & -\dfrac{\partial}{\partial x} \end{pmatrix} dy. \tag{42}$$

and the "boundary" operator on $\partial\Omega$ given by

$$\Lambda^* \mathbf{s} = \begin{cases} \sigma n_x & \text{at } x = 0, L \text{ in } x \text{ direction}, \\ \tau n_x & \text{at } x = 0, L \text{ in } y \text{ direction}, \\ \tau n_y & \text{at } y = \pm h. \end{cases} \tag{43}$$

The feasible set $\mathscr{V}_a \subset \mathscr{V}$ is a subspace incorporating the *essential* boundary conditions. For example, if the beam is clamped at both ends, then this space can be given as

$$\mathscr{V}_a = \left\{ (u, w)^T \in \mathscr{V} \mid u = w = 0 \text{ at } x = 0, L \right\}.$$

Thus for any given $\mathbf{u} \in \mathscr{V}_a$, the duality relation $\langle \Lambda \mathbf{u}, \mathbf{s} \rangle = (\mathbf{u}, \Lambda^* \mathbf{s}) = (\mathbf{u}, \mathbf{p})$ gives equilibrium conditions for this extended beam theory:

$$\Lambda^* \mathbf{s} = \mathbf{p} \quad \Rightarrow \quad \begin{cases} \dfrac{\partial}{\partial x}\sigma + \dfrac{\partial}{\partial y}\tau = 0 & \forall (x, y) \in \Omega, \\ -\displaystyle\int_{-h}^{h} \dfrac{\partial \tau}{\partial x}\, dy = p(x) & \forall x \in [0, L], \\ \tau(x, \pm h) = \pm q^{\pm}(x) & \forall x \in [0, L]. \end{cases} \tag{44}$$

Assuming that the shape of obstacle $\psi(x)$ is a strictly concave function, for the given external load $\bar{\mathbf{p}}$, the external complementarity condition for this contact problem is the well-known Signorini condition:

$$\begin{gathered} w(x) - \psi(x) \geq 0, \qquad \bar{p}(x) - p(x) \leq 0, \\ (w(x) - \psi(x))(\bar{p}(x) - p(x)) = 0 \qquad \forall x \in [0, L]. \end{gathered} \tag{45}$$

For any solution $\mathbf{u}(x, y) = (u(x, y), w(x))^T$ and $\mathbf{s}(x, y) = (\sigma(x, y), \tau(x, y))^T$, the contact region $\mathscr{Z} \subset [0, L]$ and its boundary can be defined by

$$\mathscr{Z} := \{ x \in [0, L] \mid w(x) = \psi(x) \ \forall x \in [0, L] \},$$

$$\partial \mathscr{Z} := \left\{ x \in [0, L] \mid w(x) = \psi(x), \ \int_{-h}^{h} \frac{\partial \tau}{\partial x}\, dy + \bar{p}(x) = 0 \right\}.$$

This boundary is unknown until the problem is solved. It is called the free boundary.

For a lubricated contact problem, the lubrication implies that $q^-(x) = 0$. In the case of Coulomb's law of dry friction, we have

$$|q^-(x)| < -\nu\bar{p} \quad \Rightarrow \quad u(x, -h) = 0 \qquad \forall x \in \mathscr{Z},$$
$$|q^-(x)| = -\nu\bar{p} \quad \Rightarrow \quad u(x, -h) = -\mu q^-(x)$$
$$\forall x \in \mathscr{Z} \text{ for some } \mu \geq 0.$$

Here ν is the coefficient of friction. For more general frictional contact problems, we can define a convex, l.s.c. functional $j(\mathbf{u}) = j(u(x, -h))$ such that the frictional law can be given as (cf., e.g., [31, 32]):

$$-q^-(x) = \tau(x, -h) \in \partial j(\mathbf{u}) \qquad \forall x \in [0, L].$$

This subdifferential relation is understood pointwise. It can also be written in the inverse form

$$u(x, -h) \in \partial j^*(\mathbf{s}) \qquad \forall x \in [0, L],$$

where $j^*(\mathbf{s}) = j^*(\tau(x, -h))$ is the conjugate function of j.

When a certain combination of stresses in the beam results in strains exceeding the limit of elastic behavior, plastic deformation begins to take place. Let $\Omega_e \subset \Omega$ be the elastic zone and $\Omega_p \subset \Omega$ be the plastic zone such that $\Omega_e \cup \Omega_p = \Omega$, $\Omega_e \cap \Omega_p = \emptyset$. In elastic zone $\Omega_e \subset \Omega$, the constitutive equation is given by Hooke's law,

$$\mathbf{s} = \begin{pmatrix} \sigma \\ \tau \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} \epsilon \\ \gamma \end{pmatrix} = \mathbf{Ce}, \tag{46}$$

where E, G are positive elastic constants. However, in the plastic zone $\Omega_p \subset \Omega$, the strain vector $\mathbf{e}$ can be split into two part: $\mathbf{e} = \mathbf{e}^e + \mathbf{e}^p$, i.e., the elastic strain $\mathbf{e}^e$, which is given by Hooke's law, and the plastic strain $\mathbf{e}^p$, which is given by *Hencky's plastic constitutive law*:

$$\mathbf{e}^p = \lambda \frac{\partial g^*(\mathbf{s})}{\partial \mathbf{s}} \qquad \text{in } \Omega_p,$$
$$\text{s.t. } \lambda \geq 0, \qquad g^*(\mathbf{s}) \leq 0, \qquad \lambda g^*(\mathbf{s}) = 0 \qquad \text{a.e. in } \Omega. \tag{47}$$

Here λ is a plastic flow factor; $g^*(\mathbf{s})$ is the plastic yield function,

$$g^*(\mathbf{s}) = \|\mathbf{s}\|_\alpha - \sigma_b = \sqrt{\sigma^2 + \alpha\tau^2} - \sigma_b, \tag{48}$$

σ_b is a material constant, and $\alpha > 0$ is a parameter. For the von Mises material, $\alpha = 3$; for the Tresca material, $\alpha = 4$ (cf., e.g., [3]). $\|\mathbf{s}\|_\alpha =$

$\sqrt{\sigma^2 + \alpha\tau^2}$ is the effective stress, which is a norm of the stress vector $\mathbf{s}$. For a proportional loading problem, i.e., during plastic deformation, the strain vectors $\mathbf{e}^{\mathrm{e}}$ and $\mathbf{e}^{\mathrm{p}}$ stay in the same direction, it is easy to find that

$$\lambda = g(\mathbf{e}) = \|\mathbf{e}\|_{1/\alpha} - E^{-1}\sigma_b = \sqrt{\epsilon^2 + \frac{1}{\alpha}\gamma^2} - E^{-1}\sigma_b, \tag{49}$$

where $\|\mathbf{e}\|_{1/\alpha} = \sqrt{\epsilon^2 + (1/\alpha)\gamma^2}$ is the norm of the strain vector. By using the step function $\delta(g^*)$, the constitutive relation can be written as (cf., e.g., [18]):

$$\mathbf{e} = \mathbf{C}^{-1}\mathbf{s} + g\frac{\partial g^*(\mathbf{s})}{\partial \mathbf{s}}\delta(g^*(\mathbf{s})) \quad \text{or} \quad \begin{cases} \epsilon = E^{-1}\sigma + g\dfrac{\partial g^*(\mathbf{s})}{\partial \sigma}\delta(g^*(\mathbf{s})), \\ \gamma = G^{-1}\tau + g\dfrac{\partial g^*(\mathbf{s})}{\partial \tau}\delta(g^*(\mathbf{s})), \end{cases} \tag{50}$$

subject to the internal complementarity conditions

$$g(\mathbf{e}) \geq 0, \qquad g^*(\mathbf{s}) \leq 0, \qquad g(\mathbf{e})g^*(\mathbf{s}) = 0. \tag{51}$$

The elastic region Ω_{e} and the plastic region Ω_{p} can be given by the internal complementarity conditions

$$\Omega_{\mathrm{e}} := \{(x, y) \in \Omega \mid g(\mathbf{e}(x, y)) < 0 \text{ or } g^*(\mathbf{s}(x, y)) < 0 \ \forall (x, y) \in \Omega\},$$

$$\Omega_{\mathrm{p}} := \{(x, y) \in \Omega \mid g^*(\mathbf{s}(x, y)) = 0 \text{ or } g(\mathbf{e}(x, y)) \geq 0 \ \forall (x, y) \in \Omega\},$$

respectively. The interface of elastoplastic region $\Gamma_{\mathrm{ep}} = \Omega_{\mathrm{e}} \cap \Omega_{\mathrm{p}}$ is defined by

$$\Gamma_{\mathrm{ep}} := \{(x, y) \in \Omega \mid g(\mathbf{e}(x, y))g^*(\mathbf{s}(x, y)) = 0\}.$$

For the given external force $\bar{\mathbf{p}}$, the obstacle $\psi(x)$ and the boundary condition set $\mathscr{U}_a$, the contact problem of this extended beam theory can be given as:

Problem 4. For the given obstacle function $\psi(x)$ and the external load $\bar{p}(x)$, $\bar{q}^+(x)$, find the displacement field $\mathbf{u}(x, y) = (u(x, y), w(x))^T \in \mathscr{U}_a$ and the stress s field $\mathbf{s} = (\sigma(x, y), \tau(x, y))^T$ such that the following equations are satisfied:

1. *Geometrical Equation*:

$$\mathbf{e} = \Lambda\mathbf{u} \quad \text{or} \quad \begin{pmatrix} \epsilon \\ \gamma \end{pmatrix} = \begin{pmatrix} \dfrac{\partial}{\partial x} & 0 \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} u(x, y) \\ w(x) \end{pmatrix}$$

2. *Constitutive Equation*:

$$\mathbf{e} = \mathbf{C}^{-1}\mathbf{s} + g\frac{\partial g^*(\mathbf{s})}{\partial \mathbf{s}}\delta(g^*(\mathbf{s})) \quad \text{or} \quad \begin{cases} \epsilon = E^{-1}\sigma + g\dfrac{\partial g^*(\mathbf{s})}{\partial \sigma}\delta(g^*(\mathbf{s})), \\ \gamma = G^{-1}\tau + g\dfrac{\partial g^*(\mathbf{s})}{\partial \tau}\delta(g^*(\mathbf{s})). \end{cases}$$

3. *Equilibrium Condition*:

$$\Lambda^*\mathbf{s} = \mathbf{p} \quad \text{or} \quad \begin{cases} \displaystyle\int_{-h}^{h} -\frac{\partial \tau}{\partial x}\,dy = p(x) & \forall x \in [0, L], \\ \dfrac{\partial}{\partial x}\sigma + \dfrac{\partial}{\partial y}\tau = 0 & \forall (x, y) \in \Omega; \end{cases}$$

$$\tau(x, h) = \bar{q}^+(x), \qquad \tau(x, -h) \in \partial j(u(x, -h)) \qquad \forall x \in [0, L].$$

4. *External Complementarity Condition*:

$$w(x) - \psi(x) \geq 0, \qquad \bar{p}(x) - p(x) \leq 0, \qquad (w - \psi)(\bar{p} - p) = 0.$$

5. *Internal Complementarity Condition*:

$$g(\mathbf{e}) \geq 0, \qquad g^*(\mathbf{s}) \leq 0, \qquad g(\mathbf{e})g^*(\mathbf{s}) = 0.$$

This is a strong nonlinear bicomplementarity problem. Its framework is shown in Fig. 3.

Let $\mathscr{K}$ denote a convex subset of stresses

$$\mathscr{K} = \{\mathbf{s} \in \mathscr{S} \mid g^*(\mathbf{s}(x, y)) \leq 0 \ \forall (x, y) \in \Omega\}, \tag{52}$$

the plastic superpotential $W_{\mathrm{p}}^*(\mathbf{s})$ can be defined by

$$W_{\mathrm{p}}^*(\mathbf{s}) = \begin{cases} 0, & \text{if } \mathbf{s} \in \mathscr{K}, \\ +\infty & \text{otherwise}, \end{cases} \tag{53}$$

which is the indicator function of the convex set $\mathscr{K}$. Its subdifferential is a *point to set* map

$$\partial W_{\mathrm{p}}^*(\mathbf{s}) = \begin{cases} g\dfrac{\partial g^*(\mathbf{s})}{\partial \mathbf{s}}, & \text{if } g^*(\mathbf{s}) = 0, g \geq 0, \\ \{0\}, & \text{if } g^*(\mathbf{s}) < 0, \\ \emptyset, & \text{if } g^*(\mathbf{s}) > 0. \end{cases} \tag{54}$$

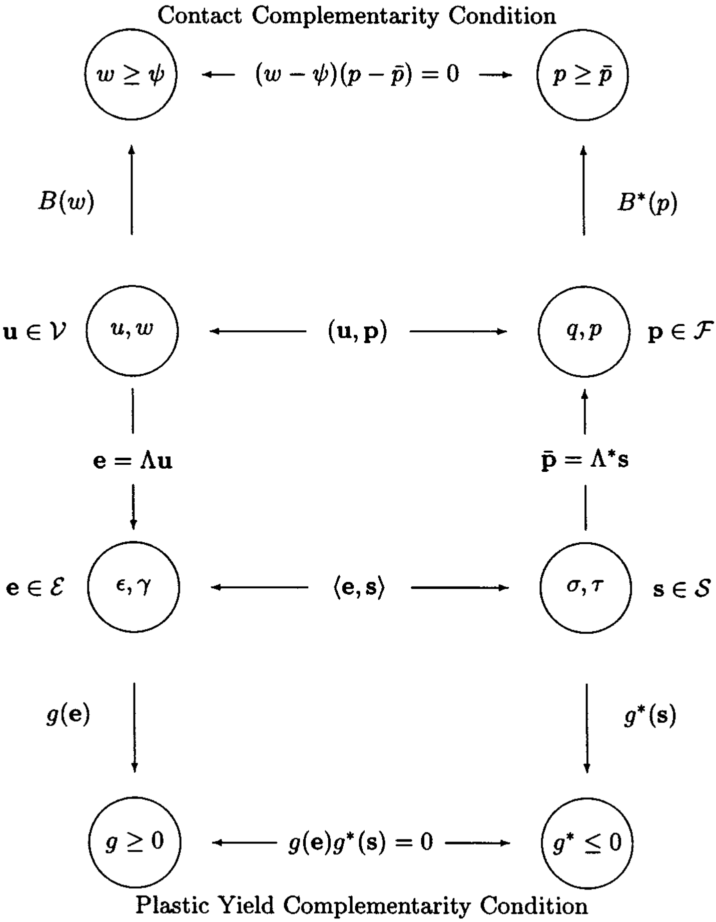

FIG. 3. Framework for contact problem of elastoplastic beam theory.

So by this subdifferentiation notation, the plastic flow law (47) can be written in the form

$$\mathbf{e}^{\mathrm{p}} \in \partial W_{\mathrm{p}}^*(\mathbf{s}). \tag{55}$$

Using the Legendre–Fenchel transformation, the conjugate function of W_p^* can be given as

$$W_{\mathrm{p}}(\mathbf{e}) = \sup_{\mathbf{s} \in \mathscr{K}} \langle \mathbf{s}, \mathbf{e}^{\mathrm{p}} \rangle = \begin{cases} 0, & \text{in } \Omega_{\mathrm{e}}, \\ \int_{\Omega} \sigma_b \left(\sqrt{\epsilon^2 + (1/\alpha)\gamma^2} - E^{-1}\sigma_b \right) d\Omega & \text{in } \Omega_{\mathrm{p}}, \end{cases}$$

which is called the support function of the convex set $\mathscr{K}$. The elastic potential is a quadratic functional:

$$W_e(\mathbf{e}) = \frac{1}{2}\int_\Omega \left[\mathbf{e}^T\mathbf{C}\mathbf{e}\,\delta^c(g(\mathbf{e})) + E^{-1}\sigma_b^2\delta(g(\mathbf{e}))\right] d\Omega. \tag{56}$$

The superpotential in this problem is

$$W(\mathbf{e}) = W_e(\mathbf{e}) + W_p(\mathbf{e}).$$

So the constitutive equation and the internal complementarity condition can be written in the subdifferential form

$$\mathbf{s} \in \partial W(\mathbf{e}). \tag{57}$$

The external potential energy $F\colon \mathscr{V} \to \mathbf{R}^{\ominus}$ can be defined as

$$F(\mathbf{u}) = (\mathbf{u}, \bar{\mathbf{p}}) - \Psi_{\mathscr{V}_a}(\mathbf{u}) - j(\mathbf{u})$$

$$= \begin{cases} \displaystyle\int_0^L [\,\bar{p}w + \bar{q}^+ u(x,h)]\,dx - j(u(x,-h)), & \\ \quad \text{if } \mathbf{u} = (u,w)^T \in \mathscr{V}_a, & \\ -\infty, & \text{otherwise.} \end{cases} \tag{58}$$

So the total potential energy $P(\mathbf{u}) = W(\Lambda\mathbf{u}) - F(\mathbf{u})$ is a convex, l.s.c. functional. The convex cone $\mathscr{C}$ is

$$\mathscr{C} = \left\{\mathbf{u} = (u,w)^T \in \mathscr{V} \mid w(x) \geq \psi(x)\ \forall x \in [0, L]\right\}. \tag{59}$$

By the Lemma 1, the bi-complementarity problem for this extended beam theory is equivalent to the following primal variational problems: Find $(u,w)^T \in \mathscr{C}$ such that

$$(\mathrm{PVP}) \qquad P(u,w) = \inf P(v,z) \qquad \forall (v,z)^T \in \mathscr{C}. \tag{60}$$

The associated variational inequality problem (PVI) has a very complicated form:

$$(\Lambda^*\partial W(\Lambda\mathbf{u}), \mathbf{v} - \mathbf{u}) + j(\mathbf{v})$$

$$\geq \int_0^L [\,\bar{p}(z - w) + \bar{q}^+(v(x,h) - u(x,h)]\,dx + j(u(x,-h))$$

$$\forall \mathbf{v} = (v,z)^T \in \mathscr{V}_a \cap \mathscr{C}. \tag{61}$$

Since the superpotential W is a nonsmooth function, the primal variational approaches for solving this bi-complementarity problem are very difficult. Now let us consider the dual approaches.

The dual feasible set $\mathscr{S}_a = \mathscr{E}_a^*$ in this problem is

$$\mathscr{S}_a = \left\{ \mathbf{s} = (\sigma, \tau)^T \in \mathscr{S} \,\middle|\, \frac{\partial \sigma}{\partial x} + \frac{\partial \tau}{\partial y} = 0 \,\forall (x, y) \in \Omega, \right.$$

$$\left. \tau(x, h) = \bar{q}^+(x) \,\forall x \in [0, L] \right\}.$$

$\mathscr{D}^* \subset \mathscr{S}_a$ in this problem is a so-called *statically admissible space*:

$$\mathscr{D}^* := \left\{ \mathbf{s} = (\sigma, \tau)^T \in \mathscr{S}_a \,\middle|\, \int_{-h}^{h} \frac{\partial \tau}{\partial x} \, dy - \bar{p}(x) \le 0 \,\forall x \in [0, L] \right\}. \quad (62)$$

By the Legendre–Fenchel transformation, the external complementary energy $F^*\colon \mathscr{S} \to \mathbf{R}^{\ominus}$ can be obtained as

$$F^*(\Lambda^* \mathbf{s}) = \begin{cases} \displaystyle\int_0^L - \psi(x) \left[\int_{-h}^{h} \frac{\partial \tau}{\partial x} \, dy + \bar{p}(x) \right] dx - j^*(\tau), \\ \quad \text{if } \mathbf{s} = (\sigma, \tau)^T \in \mathscr{D}^*, \\ -\infty, \qquad \text{otherwise}. \end{cases} \quad (63)$$

The internal complementary energy $W^*\colon \mathscr{S} \to \mathbf{R}^{\oplus}$ is a convex, l.s.c. function

$$W^*(\mathbf{s}) = W_e^*(\mathbf{s}) + W_p^*(\mathbf{s}) = \begin{cases} \displaystyle\int_\Omega \frac{1}{2} \mathbf{s}^T \mathbf{C}^{-1} \mathbf{s} \, d\Omega, & \text{if } \mathbf{s} \in \mathscr{K}, \\ +\infty, & \text{otherwise}. \end{cases} \quad (64)$$

On the space $\mathscr{D}^* \cap \mathscr{K}$, the total complementary energy functional $P^*(\mathbf{s})$ can be simply written as

$$P^*(\sigma, \tau) = \int_0^L - \psi(x) \left[\bar{p}(x) + \int_{-h}^{h} \frac{\partial \tau}{\partial x} \, dy \right] dx$$

$$- \int_\Omega \frac{1}{2} [E^{-1} \sigma^2 + G^{-1} \tau^2] \, d\Omega - j^*(\tau). \quad (65)$$

Thus the dual variational problem for this nonlinear bi-complementarity problem can be given as the following: Find $\bar{\mathbf{s}} = (\bar{\sigma}, \bar{\tau})^T$ such that

$$(\text{DVP}) \qquad P^*(\bar{\sigma}, \bar{\tau}) \ge P^*(\sigma, \tau) \qquad \forall (\sigma, \tau)^T \in \mathscr{D}^* \cap \mathscr{K}. \quad (66)$$

The dual variational inequality for this problem has a very simple form:

(DVI)

$$\int_\Omega \left[E^{-1}\bar{\sigma}(\sigma - \bar{\sigma}) + G^{-1}\bar{\tau}(\tau - \bar{\tau}) \right] d\Omega + j^*(\tau)$$
$$\geq \int_\Omega \psi(x) \frac{\partial}{\partial x}(\bar{\tau} - \tau)\, d\Omega + j^*(\bar{\tau}) \qquad \forall (\sigma, \tau)^T \in \mathscr{D}^* \cap \mathscr{K}. \tag{67}$$

For the lubricated contact problem, the dual problems (DVP) and (DVI) are quadratic programmings on the closed convex subset $\mathscr{D}^* \cap \mathscr{K}$. Since $W^*\colon \mathscr{S} \to \mathbf{R}$ is a strictly convex functional, the solution of the dual problems is unique. By Theorem 1, we know that both dual problems are equivalent to the primal problems and BCP. However, compared with primal problems (60) and (61), the dual problems (66) or (67) are much easier to solve. In numerical analysis, the dual approaches will provide the lower bound solutions. Moreover, using the linear equilibrium constraint in the statically admissible space $\mathscr{D}^*$, the degrees of freedom in nonlinear iteration can be reduced via the mixed finite element method given in [11]. The complementary finite element method for fully nonlinear, nonsmooth variational problems was discussed in [13, 14].

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